UNIVERSITY OF LIVERPOOL

JANUARY EXAMINATIONS 2012

Bachelor of Science: Year 3
Master of Physics: Year 3
Master of Physics: Year 4

STATISTICAL AND LOW TEMPERATURE PHYSICS

TIME ALLOWED: 3 hours

INSTRUCTIONS TO CANDIDATES

Answer <u>all</u> questions.

Question 1 carries 50% of the total marks.

Questions 2 and 3 each carry 25% of the total marks.

Answer either part (a) or part (b) of questions 2 and 3.

In the event of a student answering both parts of an either/or question and not clearly crossing out one answer, only the answer to part (a) of the question will be marked.

The marks allotted to each part of a question are indicated in square brackets.

All symbols have their usual meanings unless otherwise stated.

Question 1.

(a) One mole of a spin ½ salt sits in a 1 T magnetic field, at a temperature of 1 K.

i) Find the energies of the magnetic levels. **[2]**

ii) Write down the formulae for the most probable macrostate. **[2]**

iii) Calculate the Boltzmann factor for each level. **[2]**

iv) What is the ratio of the populations in the two levels? **[2]**

v) Find the populations (i.e. the number of moles in each level). **[2]**

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Solution

1(a)

(i) The energies are $-\mu_B B$ and $+\mu_B B$.

$\mu_B B = 9.27 \times 10^{-24} \times 1 = 9.27 \times 10^{-24}$ J [U2]

(ii) $n_i = A \exp(-\varepsilon_i/k_B T)$

Lower state: $n_1 = A \exp(+\mu_B B//k_B T)$

Higher state: $n_2 = A \exp(-\mu_B B//k_B T)$

Where A is a constant [U2]

(iii) Boltzmann factor is $\exp(-\varepsilon_i/k_B T)$

$\mu_B B//k_B T = 9.27 \times 10^{-24} / 1.38 \times 10^{-23} = 0.6717$

Lower state: $\exp(+\mu_B B//k_B T) = 1.9576$

Higher state: $\exp(-\mu_B B//k_B T) = 0.5108$ [U2]

(iv) $n_1 : n_2 = \exp(+\mu_B B//k_B T) : \exp(-\mu_B B//k_B T) = 3.8324 : 1$ [U2]

(v)

Population in lower state = $n_1 / (n_1 + n_2)$ x 1 mole = 0.7931 mol

Population in higher state = $n_2 / (n_1 + n_2)$ x 1 mole = 0.2069 mol [U2]

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(b) One mole of Neon (relative atomic mass = 20) gas at 300 K is in a cube of side L.

i) Find the average energy of the atoms. **[2]**

ii) Find the corresponding wavevector. **[3]**

iii) If L is 30 cm, find the spacing between wavevectors. **[2]**

iv) Why do we need density of states? **[3]**

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Solution

1(b)

(i) Assuming that Neon is an ideal gas, the average energy is $3k_BT/2 = 1.242 \times 10^{-20}$ J

[U2]

(ii) $E = \dfrac{\hbar^2 k^2}{2m}$

E is the average energy above.

k is wavevector

m is the mass of Neon atom = $20m_u$

$k = \dfrac{\sqrt{2mE}}{\hbar}$ = 2.735×10^{11} m^{-1}. [U3]

(iii) The spacing is $\pi/L = \pi/0.3 = 10.472$ m^{-1}. [U2]

(iv)

Density of states help us to find the sums for number of particles and total energy when the number of energy levels is very large.

This happens when the average energy of the particles is large compared with the spacing between levels. [U3]

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(c) A spin ½ salt sits in a magnetic field. At high temperatures, there are 0.1 mole of spin ½ ions at each of the two magnetic energy levels.

i) At 1 K, the number of the ions at the lower level increased by 50%. Find the population at each level. **[3]**

ii) Find the energy difference between the levels. **[3]**

iii) How much heat is given out? **[4]**

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Solution

1(c)

(i) At high T,

Population at lower level = 0.1 mol

Population at higher level = 0.1 mol

At 1 K, Population at lower level $n_1 = 0.1 + 0.05 = 0.15$ mol

So, population at lower level $n_2 = 0.1 - 0.05 = 0.05$ mol [U3]

(ii)

Let the energies of the levels be ε_1 and ε_2.

The macrostate is Boltzmann distribution.

So $n_1 = 0.15 \text{ mol} = \exp(-\varepsilon_1/k_BT)$

and $n_1 = 0.05 \text{ mol} = \exp(-\varepsilon_2/k_BT)$

Dividing gives $3 = \exp((\varepsilon_2 - \varepsilon_1)/k_BT)$

Solving, the energy difference is $\varepsilon_2 - \varepsilon_1 = k_BT \ln 3 = 1.5161 \times 10^{-23}$ J [U3]

(iii)

From above, 0.05 mol of ions moved from higher to lower energy.

This is a number of $0.05\ N_A$. [U2]

The energy of each ion falls by $\varepsilon_2 - \varepsilon_1$. This is given out as heat.

So the total heat given out is $0.05\ N_A\ (\varepsilon_2 - \varepsilon_1) = 0.4565$ J [U2]

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(d)

i) The presence of a magnetic field in a macroscopic wavefunction of electrons must produce a current. How does this explain the Meissner's effect? **[2]**

ii) Discuss how this gives rise to London's penetration depth. **[4]**

iii) Sketch the heat capacity versus temperature graph for a superconductor, above and below the transition temperature. How does this suggest the existence of an energy gap? **[4]**

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Solution

1(d)

(i)

When an external field is present, the wavefunction produces a current. This gives a magnetic field that opposes the applied field. So the external field in the superconductor can be cancelled, or expelled. [B2]

(ii)

The current produced is proportional to the charge density of electrons.

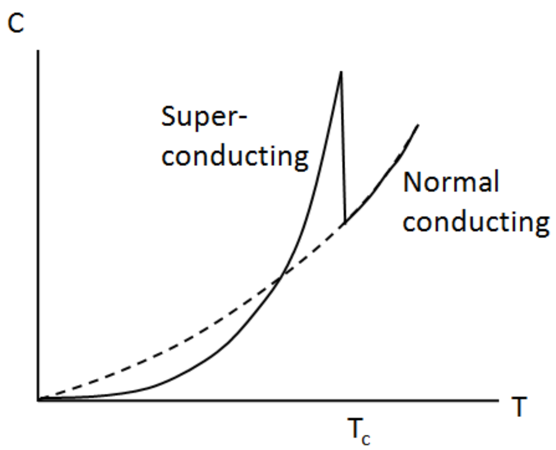
Even if this density is very large, the external field would never be completely expelled.[B2]

The reason is that some current is needed to produce the opposing field.

In order for current to exist, some field must penetrate into the superconductor.

The depth of this penetration is the London's penetration depth. [B2]

iii)



[B2]

Above transition, the heat capacity follows that of the normal metal, $c_v = AT + \gamma T^3$.

Below transition, it changes to $c_v = B\exp(-\Delta/k_BT)$, which has the same form as a Boltzmann factor between two levels of energy gap Δ. [B2]

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(e) A body can move with zero resistance through superfluid ^{4}He.

i) What excitations are possible in superfluid ^{4}He? **[2]**

ii) Consider an excitation of energy E and momentum p. The body's velocity must be above E/p before excitation is possible. What are the conservation laws that lead to this? **[2]**

iii) Sketch the dispersion relation of the excitations in superfluid ^{4}He. Draw the line with gradient equal to the minimum E/p. **[4]**

iv) Why does the body experience no resistance when its velocity is below the minimum E/p? **[2]**

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<u>Solution</u>

1(e)

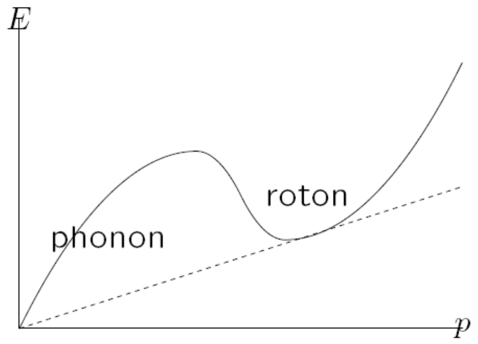
(i) Phonons.

rotons. [B2]

(ii) Conservation of energy

Conservation of momentum [B2]

(iii)



[B2]

Near p = 0, excitations are due to phonons, which have nonzero gradient.

The gradient of the dashed line is E/p.

It cannot be zero because the roton minimum is above zero. [B2]

(iv)

For velocity below the minimum E/p, no excitation is possible.

If there is no excitation, the body cannot lose energy, so it experiences zero resistance. [B2]

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Question 2. Answer **<u>either</u>** (a) **<u>or</u>** (b)

2(a)

One mole of copper is at 1 K. Each atom supplies two conduction electrons.

i) Assuming that the electrons behave like an ideal gas, write down the expression for the average kinetic energy of the electrons. Hence find the heat capacity. **[3]**

ii) At 1 K, the measured heat capacity is 0.6 mJ/K. Explain why it is different from the answer in (i). **[2]**

iii) With the help of a graph, estimate the energy range of the electrons that are excited above the Fermi energy at temperature T. **[4]**

iv) Using the density of states for the ideal gas,

$$g(\varepsilon) = \frac{4m\pi V}{h^3}\sqrt{2m\varepsilon}\ ,$$

derive an expression for the number, n, of excited electrons. (Molar volume of copper is 7.11 cm^3). **[4]**

v) Why is it reasonable to assume that the excited electrons in (iv) behave like the ideal gas? **[4]**

vi) Derive an expression for the heat capacity of the excited electrons using the ideal gas formula from (i). **[4]**

vii) Find the Fermi energy. Using the expression from (vi), calculate the heat capacity for copper at 1 K. Compare with values in (i) and (ii). What does this suggest? **[4]**

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<u>Solution</u>

2(a)

(i) The average kinetic energy is $\varepsilon = 3k_BT/2$. [U1]

There are 2 conduction electrons per atom, so there are $2N_A$ conduction electrons in total.

The total energy $U = 2N_A\varepsilon = 2N_A \times 3k_BT/2 = 3RT$.

The heat capacity $C = dU/dT = 3R = 24.93$ J/K [U2]

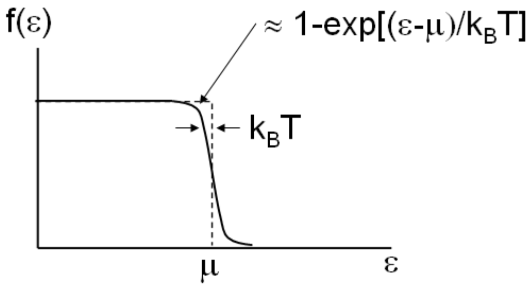
(ii) The electrons do not behave like ideal gas at all.

In the electron gas, the electrons are stacked up in energy levels, from the ground state to a maximum energy. When heated, only a small fraction of the electrons near the top can be excited. As a result, the heat capacity is much smaller. [U2]

(iii) The probability that a state at energy ε is given by the Fermi-Dirac distribution

$$f(\varepsilon) = 1 / (\exp[(\varepsilon - \mu)/(k_B T)] + 1).$$

At 1 K, the graph is quite close to the step at the Fermi energy $\mu = E_F$.



This is because the Fermi energy for a metal is usually much higher than $k_B T$. [U2]

So for energy a few $k_B T$ smaller than μ, the exponential function quickly becomes small. Then

$$f(\varepsilon) \approx 1 - \exp[(\varepsilon - \mu)/(k_B T)].$$

When energy falls below μ, the probability moves towards 1 exponentially. It falls roughly to 1/e after $k_B T$.

So $k_B T$ is approximately the range of energy of the electrons that are excited. [U2]

(iv) The density of states is $2 \times g(\varepsilon) = 2 \times (4m\pi V/h^3)(2m\varepsilon)^{1/2}$.

At the Fermi energy, this is $2g(E_F)$. [U2]

The number of excited electrons is the number of states in their energy range, which is $k_B T$.

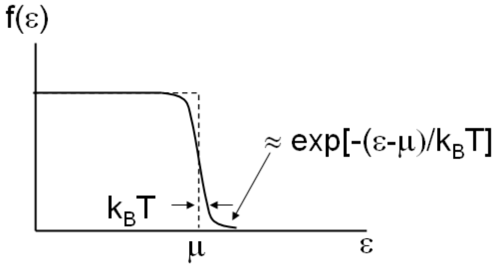
This number n of the states is $2g(E_F) \times k_B T$. [U2]

(v) For energy above E_F, the exponential term in

$$f(\varepsilon) = 1 / (\exp[(\varepsilon - \mu)/(k_B T)] + 1).$$

quickly gets large, so that

$f(\varepsilon) \approx \exp[-(\varepsilon - \mu)/(k_B T)]$. [U2]



This is just the Boltzmann distribution, the same as that of an ideal gas. [U2]

(vi) The average energy of an ideal gas particle is $3k_BT/2$.

Since the excited electrons behave like an ideal gas, the total energy is $U = n \times 3k_BT/2$. [U2]

Using the previous expression for n, $U = 2g(E_F) \times k_BT \times 3k_BT/2$

$= 3g(E_F)k_B^2T^2$

Heat capacity $C = dU/dT = 6g(E_F)k_B^2T$ [U2]

(vii) $N = 2N_A$. The Fermi energy is $E_F = (\hbar^2/2m)(3\pi^2N/V)^{2/3} = 1.789 \times 10^{-18}$ J.

Substituting, we find $C = 6g(E_F)k_B^2T = 0.5777$ mJ/K. [U2]

This is fairly close to the measured 0.6 mJ/K, within 4%.

It is strong evidence that both the Fermi-Dirac distribution, and the ideal gas model of the excited electrons, are correct. [U2]

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2 (b)

Dilution cooling makes use of a mixture of liquid ^{3}He in liquid ^{4}He.

i) Sketch a phase diagram of the mixture. Label the axes, the lambda line and the phase separation region. **[5]**

ii) Describe qualitatively what happens when the temperature of a 50% mixture falls just below the phase separation curve. **[5]**

iii) Describe qualitatively what happens when the temperature of this mixture reaches a few milliKelvin. **[3]**

iv) Sketch a diagram to explain qualitatively how the ^{3}He could be removed from the bottom layer. **[6]**

v) Why is this necessary? What would then happen to the top layer, and what must we do about it? **[6]**

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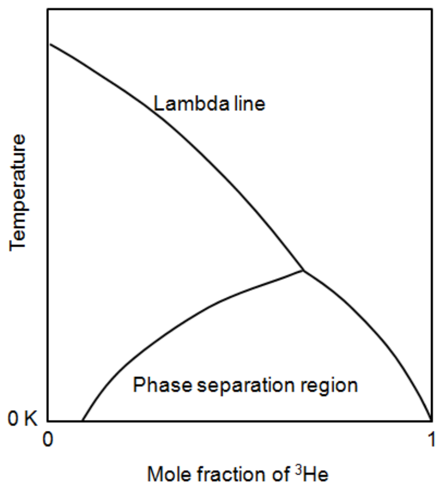
Solution

2(b)

(i)



For graph [B2]

For labels [B3]

(ii)

At a temperature in the phase separation region, the concentration of ^{3}He that is in the region is not possible.

Instead, the mixture would separate into two layers. [B2]

One layer has a higher concentration of ^{3}He than the other layer.

The layer with higher concentration of ^{3}He is on top, because 3He atom is lighter than 4He atom.

[B3]

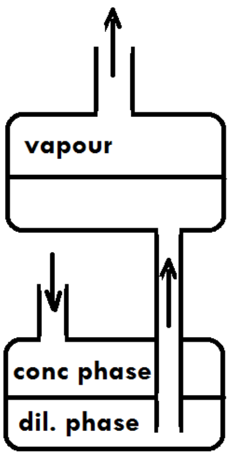
(iii)

The mixture separates into two layers.

The top layer is nearly pure ^{3}He.

The bottom layer reaches a small but nonzero fraction of ^{3}He in the mixture. [B3]

(iv)



[B3]

A tube leads from the bottom layer to a chamber higher up.

Pressure on the top layer forces the bottom layer up to the higher chamber.

There, the dilute phase vaporises.

The vapour contains a higher concentration of ^{3}He, because

^{3}He has higher vapours pressure.

In this way, ^{3}He is removed from the bottom layer. [B3]

(v)

Cooling stops when the dilute phase is saturated with ^{3}He at 6.6%.

^{3}He has to be removed to allow more ^{3}He to go from top to bottom layer,

so that cooling can continue. [B3]

The volume of the top layer would decrease as ^{3}He moves to the bottom layer.

^{3}He must be replaced.

This is done by liquefying the ^{3}He that has been removed from the higher chamber,

and passing it back down to the top layer. [B3]

Question 3. Answer **either** (a) **or** (b)

(a)

i) Why is the Bose-Einstein condensate a good candidate for explaining superfluidity and superconductivity? **[5]**

ii) Using a sketch of the Fermi-Dirac distribution graph, explain what happens to the chemical potential μ of a boson gas as temperature falls to 0 K? **[5]**

iii) In terms of the density of states g(ε),where ε is the energy, write down the expression for the number of particles N. Explain when and why the number of excited bosons may be written as

$$N_{ex} = \int_0^\infty \frac{g(\varepsilon)d\varepsilon}{\exp(\varepsilon / k_B T) - 1},$$

where T is the temperature. **[5]**

iv) Given that the solution is

$$N_{ex} = \left(\frac{2\pi m k_B T}{h^2}\right)^{3/2} 2.612V,$$

where V is the volume that contains the N particles, explain how to find the condensation temperature T_{BE}. Determine the value of T_{BE} for liquid ^{4}He, with molar volume 27.58 cm^3. **[5]**

v) In terms of g(ε), write down the expression for the energy U below T_{BE}. Given that

$$U = 0.7704 k_B N \frac{T^{5/2}}{T_{BE}^{3/2}}$$

is the solution, derive the heat capacity C. Calculate the value of C for one mole of ^{4}He at T_{BE}, and sketch the graph of C versus T. **[5]**

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Solution

3(a)

(i)

BEC is a macroscopic wavefunction. The number of particles condensed to the ground state is a significant fraction of the bulk total.

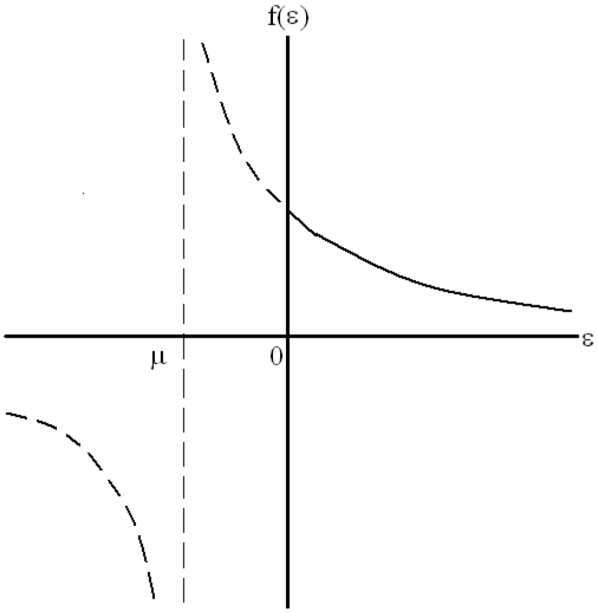
The wavefunction is in one energy state. The next higher state is separated by some amount of energy. If there is not enough energy, the particles cannot get excited. [B2]

Atoms in a superfluid flow round an obstacle without any viscosity. Electrons in a superconductor flow past impurities without any resistance.

This is possible if the atoms or electrons form a macroscopic wavefunction. If the flow is slow, there is not enough energy to excite the particle. Then there is no energy loss, so no resistance.[B2]

Since the BEC is a macroscopic wavefunction, this makes it a good candidate. [B1]

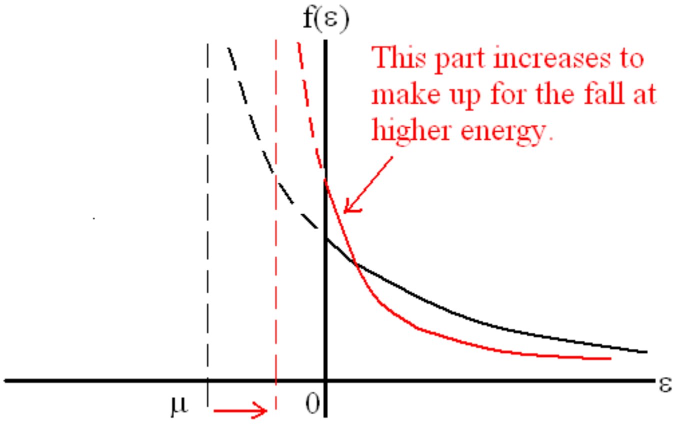
(ii)



Energy is positive. So $\mu < 0$, else $f(\varepsilon)$ can be negative. [B2]

As $T \rightarrow 0$, $f(\varepsilon) = 1/[\exp((\varepsilon - \mu)/k_B T)-1]$ falls. [B1]

So that $N = \int g(\varepsilon)f(\varepsilon)d\varepsilon$ remains the same, μ must $\rightarrow 0$



[B2]

(iii)

$$N = \int_0^\infty \frac{g(\varepsilon)d\varepsilon}{\exp((\varepsilon - \mu)/k_B T) - 1}$$ [U1]

When T falls, $\mu \rightarrow 0$ so that integral remain = N.

When $\mu = 0$, T is defined as T_{BE}.

μ cannot become positive. If T falls below T_{BE}, then integral becomes < N. [U2]

Define $N_{ex} = \int_0^\infty \frac{g(\varepsilon)d\varepsilon}{\exp(\varepsilon/k_B T)-1}$.

Interpretation: $N-N_{ex}$ condensed to ground state. So N_{ex} is excited electrons. [U2]

(iv)

At T_{BE}, the particles just start condensing.

So N_{ex} = N the total. [U2]

So can find T_{BE} using

$$N = \left(\frac{2\pi m k_B T_{BE}}{h^2}\right)^{3/2} 2.612V$$ [U1]

Use N = N_A and V = 27.58 cm^3, solve for T_{BE}, get 3.144 K. [U2]

(v)

$$U = \int_0^\infty \frac{\varepsilon g(\varepsilon)d\varepsilon}{\exp(\varepsilon/k_B T)-1}$$ [U1]

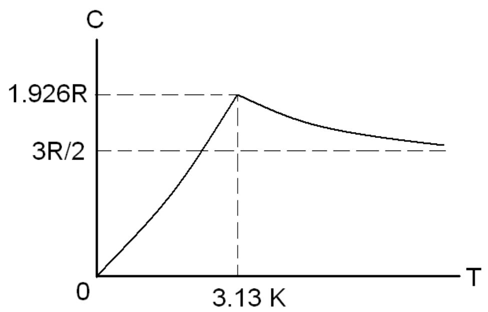
C = dU/dT = $0.7704 k_B N \frac{(5/2)T^{3/2}}{T_{BE}^{\ 3/2}}$

Use N = N_A, T = T_{BE}.

Find C = $0.7704 k_B N_A (5/2)$ = 1.926 R. [U2]

T → 0, C → 0.

T → large, C → ideal gas result 3R/2.



[U2]

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3 (b)

Superconductivity in a metal is explained by the forming of Cooper pairs. A Cooper pair is a pair of electrons that for some reason are able to attract each other.

i) Describe one experimental result that is consistent with such a binding energy. **[5]**

ii) Explain qualitatively one experiment which suggests that superconductivity has something to do with lattice vibration. **[5]**

iii) Sketch a picture to explain how movements of the atoms could produce an attraction between two electrons. **[5]**

iv) Even if this attraction is possible, it would be very weak. Using the Fermi level, explain why the electrons do not just escape come apart. **[5]**

v) How is the forming of Cooper pairs consistent with the macroscopic wavefunction that is used to explain the Meissner's effect? **[5]**

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<u>Solution</u>

3(b)

(i)

One experiment is measurement of heat capacity of a superconductor.

A graph of heat capacity against temperature shows a sudden jump at the transition temperature.

Above transition temperature, heat capacity comes from electrons and phonons. [B2]

Below, it follows $\exp(-\Delta/k_BT)$

This is like Boltzmann distribution for two energy levels.

So a binding energy suggests two levels – one for bound state, one when the bond is broken. [B3]

(ii) One experiment is the measurement of transition temperature of mercury.

The results show that different isotopes of mercury have different transition temperature.

The difference between isotopes is the masses of the atoms and the number of neutrons.

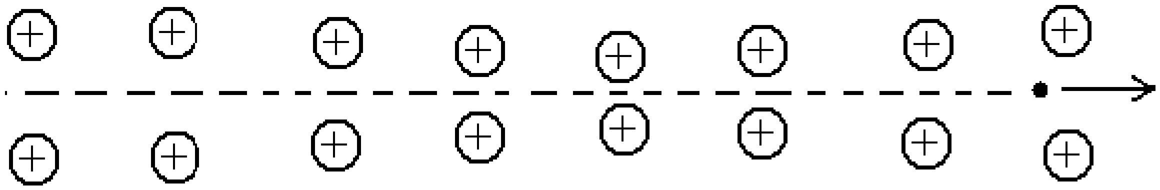
The neutrons do not interact directly with the electrons. [B2]

Movements of atoms do. We know this from effect of heat on conduction.

Heavier atoms move more slowly.

This suggests lattice vibration is an effect on superconductivity. [B3]

(iii)



[B3]

When a conduction electron moves in between the positive ions, it attracts them.

So the ions come closer.

This produces a region of higher positive charge.

This region in turn attracts other electrons. [B2]

(iv)

The two electrons in the Copper pair come from the Fermi level.

When they form the Cooper pair, their energies decrease slightly.

This decrease is the binding energy from the attraction.

This means that the energy of each electron in the pair is now slightly lower than the Fermi energy.

We would expect the electrons to escape because their kinetic energies are still much larger than the binding energy. [B2]

However, if they just come apart, their energies would still be below the Fermi level, where all states are already occupied.

This is not allowed by the exclusion principle.

So if the electrons escape from the Cooper pair, they have no place (energy states) to go to.

By staying together, they form a boson, which does not have to obey the exclusion principle.

In this way, they are forced to stay together. [B3]

(v)

The Cooper pair is a boson.

Many bosons can occupy the same state. [B2]

If all bosons go the ground state, they have the same wavefunction - in this case, a BEC.

This is consistent with the idea of a macroscopic wavefunction.

The macroscopic wavefunction has been used to explain the Meissner's effect and flux quantisation.

[B3]

CONSTANTS

Speed of light in vacuum	c	=	3.00×10^{8} ms^{-1}
Permeability of vacuum	μ_0	=	$4\pi \times 10^{-7}$ Hm^{-1}
		=	$4\pi \times 10^{-7}$ $VsA^{-1}m^{-1}$
Permittivity of vacuum	ε_0	=	8.85×10^{-12} Fm^{-1}
		=	8.85×10^{-12} $AsV^{-1}m^{-1}$
Elementary charge	e	=	1.60×10^{-19} C
Planck constant	h	=	6.63×10^{-34} Js
	$h/2\pi = \hbar$	=	1.05×10^{-34} Js
Avogadro constant	N_A	=	6.02×10^{23} mol^{-1}
Boltzmann constant	k_B	=	1.38×10^{-23} JK^{-1}
Gas constant	R	=	8.31 $JK^{-1}mol^{-1}$
Unified atomic mass constant	m_u	=	1.66×10^{-27} kg
		=	931.5 $MeVc^{-2}$
Electron mass	m_e	=	9.11×10^{-31} kg
Proton mass	m_p	=	1.67×10^{-27} kg
Gravitational constant	G	=	6.67×10^{-11} Nm^2kg^{-2}
Acceleration due to gravity	g	=	9.8 ms^{-2}
Bohr magneton	μ_B	=	9.27×10^{-24} JT^{-1}